THE NATURE OF ALGEBRAIC ABILITIES*
(Continued)

By EDWARD L. THORNDIKE
Institute of Educational Research
Teachers College, Columbia University

Ability with Problems

One of the most revered features of algebra is training in organizing a set of facts given in a problem described in words, into an equation or set of equations such that solving will produce the desired answer. This, so far as the problems are genuine ones whose answers a sane person in the real world might seek, is admirable. The genuine problems appropriate to a reasonable life do not, however, often lead to such fractional expressions, or quadratic equations, or denominators that need to be rationalized, or sequences wherein x appears three or four times, as are being studied and await "application to problems." The genuine problems are mostly of a type where x appears once, the other elements being numbers, and require only straightforward arithmetic plus certain conventions with respect to parentheses. Consequently, problems have been made up to give the pupil trainning in applying his more subtle algebraic techniques. These pervade the textbooks, courses of study and examinations. The following are samples:

The earth and a certain number of planets revolve around the sun. Twice the number of planets which are nearer to the sun than the earth is plus one equals the number of planets which are farther from the sun than the earth is. Find the number of planets nearer to and the number more remote from the sun than the earth is.

If a realroad train is made up of five sleeping cars, one parlor car and a certain sort of engine, its cost is $129,200. The cost of each sleeping car is $300 more than the cost of the engine. The cost of the parlor car is five-sixths of the remainder when the value of the engine is diminished by $100. Find the cost of the engine, the parlor car, and of a sleeping car.

The front wheel of a cart makes 16 revolutions more than the rear wheel in going 360 feet. If, however, the circumference of the front wheel were increased by a third, and that of the rear wheel by a fifth, the front wheel would make only 10 more revolutions than the rear wheel in going the same distance. Find the circumference of each wheel.

* The studies reported in this article were made possible by a grant from the Commonwealth Fund.

The hours required by Mr. A to travel a certain distance equals the number of miles he travels per hour. Mr. B goes the same distance in 2 hours less time by going three miles more per hour. Find the rate of travel and time taken by Mr. A.

There are two angles, one of which is 5° less than the other. If the number of degrees in each is multiplied by the number in its supplement, the product obtained from the larger of the given angles exceeds the other product by the square of the number of degrees in the smaller of the given angles. Find the angles.

Shall the ability to solve problems mean the ability to solve such as these, or shall it mean the ability to solve genuine problems of a sane life?

It has been customary to select and arrange the problem materials almost wholly from the point of view of the algebraic technique to be applied. The teacher or textbook maker, having taught the pupil how to operate with algebraic fractions, for example, looks about for problems which will lead to fractional equations. Other characteristics are treated as of minor importance. From the functional point of view, emphasizing ability to use algebra in solving problems which life will offer, it seems desirable to consider the lives of boys and girls and men and women as students, citizens, fathers and mothers, lawyers, doctors, business men or nurses, and select problems which they may usefully solve and which are properly solved by algebraic methods. These may then be arranged according to the technique involved if this is desirable; but from the functional point of view much is to be said in favor of arranging them with consideration also for their natural connections in the world of fact and their logical connections in the mind. Problems about public health, for example, may well be on the same page even though one should involve a simple equation without fractions, one a fractional equation, one a radical and one a quadratic.

The arrangement is of much less consequence than the choice. Most important of all is the general choice of attitude—are problems to be an exercise ground for algebraic insight, or is algebra to be a tool for life's needs?

The quantitative problems of life usually come in connection with real things, events and relations. There are real fields and floors, or at least maps and house plans; real ships to be navigated, guns to be pointed, alloys to be compounded, medicines to be diluted, electric cells to be connected. The abilities eventually desired are preferably abilities to deal with real situations.

Since it is inconvenient to provide these real situations in schools, we have recourse to verbal descriptions of them. We cannot, however, take it for granted that the ability to manage a certain quantitative problem as it is described in words is identical with the ability to manage the same problem when it actually arises in a real situation. The difficulties of the described problem may be largely linguistic; to take an extreme case, a person obviously could not solve a problem no matter how easy it was in reality, if it were put to him in an unknown language. On the other hand, the verbal description may be far more suggestive of the procedure to follow than the real situation would be. If a boy should think "In how many years shall I be half as old as my father?" and proceed to solve the problem he would have to know enough to ask, "How old is he now?" "How old am I now?"; and he might puzzle about exact birthdays, even the time of day of birth or allowance for leap years. In the described problem, "A boy is 14. His father is 40. In how many years will he be half as old as his father?" he is given all the data needed, and encouraged not to bother about anything more than getting a certain number which, if added to 40 and 14, makes one result twice the other.

In proportion as we retain the older view that the main value of problem-solving is its formal disciplinary value as a mental gymnastic, the distinction between ability with a problem as it occurs in reality and ability with a similar problem described in words approaches zero importance. We may even deliberately make the verbal description much easier or much harder to understand than the corresponding real situation would be.

This view is, however, hardly tenable with respect to problems that assume to have anything to do with real situations of business, science, technology, or the home. If problems have only formal disciplinary value as mental gymnastic, we may as well use unreal problems about the square of somebody's age being equal to the cube of half of his age less 2½ times his age, or about consecutive numbers, or about fractions whose numerators and denominators are related in divers ways. If we have problems about realities at all, we probably have them to train the pupil to manage realities themselves rather than to manage words about them.

One special difference between problems arising in connection with situations actually present to sense and the customary verbal problems of the algebra class, is that in the former needed data may be missing and irrelevant data may be present in large numbers, whereas in the latter we have practically accepted it as a rule that every problem should be solvable from the data given without any further additional data and that all the data given in the problem must be used in order to attain the solution.

There may be in use in some schools problems where the pupil has to decide whether the data are adequate for its solution and problems where he searches elsewhere for the additional data needed, but they are surely very rare. Problems where the data to be used in framing the equations have to be chosen from amongst many data irrelevant for that particular problem are almost equally rare.

Our custom in schools of making each problem a little paragraph of statements all to be used without adding or subtracting one jot or tittle, like the pieces in a picture puzzle, is so fixed that we find one of our standard textbooks announcing the following:

1. Every problem gives a relation between some unknown numbers.
2. There are as many distinct statements as there are unknown numbers.
3. Represent one of the unknown numbers by a letter; then, using all but one of the statements, represent the other unknowns in terms of that same letter.
4. Using the remaining statement, form an equation.

(Wells and Hart, '12, p. 100.)

Is problem solving in algebra to be only a puzzle game of fitting translated phrases into a proper equation?

There is, then, a wide range of possible opinion about what problem solving does mean and about what it should mean. We should surely try to make it mean something more educative than the solution of more or less attractive puzzles made up to exercise algebraic technique or to give indiscriminate practice in "thinking." We shall return to this subject in a separate article where the attempt will be made to clear up the psychology and pedagogy of problem solving in algebra.

The psychological demands and the psychological effects of organizing the facts of a problem into an equation or equations such as will, when solved, give the desired answer varies greatly

according as the problem is, so to speak, an "original" which the pupil thinks out, or follows other similar problems which he has learned to handle by special training *ad hoc.*

The approved theory has been, and still is, that the former should be the process in the main, but skilful teachers seem to think or fear that actual proficiency in solving the problems that are met in life (or at least those that are met in examinations) is best secured by special training in certain routines such as "let x equal the smaller number, when there is a choice," or "be sure to use all the numbers that are given," and by still more specialized training with rate problems, mixture problems, tank-and-pipe problems, and the like.

Rugg and Clark argue that directed practice with many different kinds of problems will make the pupil "able to use the method in solving any kind that you may happen to meet later" ('18, p. 208); and provide this directed practice with (I) Problems relating to age, (II) Problems in which a number is divided into two or more parts, (III) Problems based on coins, (IV) Problems based on relations between time, rate, and distance, (V) Problems involving percents, (VI) Problems concerning perimeters and areas, and, (VII) Problems based on levers. Their treatment of the first group is as follows:

"Section 97. *Need for tabulating the data of word problems.* Many problems involve so many different statements that it is practically *necessary to arrange the steps in the translation in very systematic tabular form.* Take an example like this:

" 'John's age exceeds James' by 20 years. In 15 years he will be twice as old as James. Find the age of each now.'

"Before we can write this statement in the form of an equation we must express in algebraic form *four* different things: (1) John's age *now;* (2) James' age *now;* (3) John's age in 15 years; and (4) James' age in 15 years. These four facts can best be stated in a table like this:

"(First step) Let n represent James' age *now.*

"(Second step) Tabulate the data.

TABLE 15

	Age Now	Age in 15 Years
John's age	$n + 20$	$n + 20 + 15$
James' age	n	$n + 15$

"With all the facts expressed in letters we can now state the equation which *tells the same thing as the original word* statement; namely:

"(Third step) $n + 20 + 15 = 2(n + 15)$

"We are now ready for the

"(Fourth step) the solution of the equation; the steps are as follows:"

Then follow explanations of solving and checking and sixteen carefully graded tasks leading up to five of the customary age problems such as, "A man is now 45 years old and his son is 15. In how many years will he be twice as old as his son?" ('18, p. 208.)

Shall we treat the problem material as a series of originals, or as a collection of typical groups of problems, each group of which the pupil learns how to solve much as he learns how to subtract a negative number or multiply x^a by x^b by adding exponents? Or shall we treat part of the problem material in the one way and the rest in the other? This last is just what we do in arithmetic. Certain groups of problems (as about areas, perimeters, discounts, insurance, compound interest, taxes and commissions) are prepared for by special training with each. Certain other problems are left to the undirected ingenuity of the pupils.

Ability with Graphs

The ability to understand, construct and use graphs needs definition in at least two respects. (A) Shall school algebra present elementary facts concerning all graphs that are simple and important, or shall it limit itself to the graphic presentation of a relation between two variables through the Cartesian coordinate system, and to such introductory matter as facilitates that? (B) Assuming the second answer to A, shall it deal with graphs of irregular relations not presentable in any equations which the pupil can be expected to manage, or shall it restrict itself to the straight line, parabola, hyperbola, and the like? Question A may be made clear and vivid in this form: "Which of the types of graphs on pages 85 and 86 shall the pupil be taught to understand, construct and use?"

The graphs which are simple and important may be classified (I) descriptions of the way in which a certain quantity is

divided; (II) general comparisons of two or more magnitudes; (III) comparisons of two or more magnitudes which are put in order by their relation to some characteristic; and (III-A) comparisons of two or more magnitudes which are themselves frequencies of occurrences and are put in order by their relation to the magnitude of some characteristic.

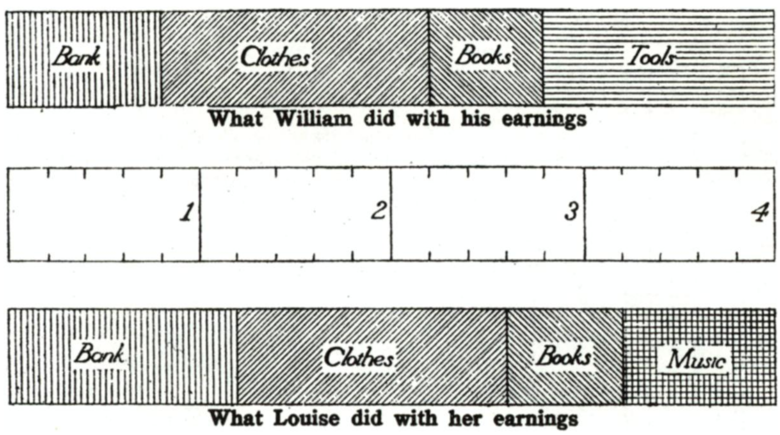


FIGURE 1

TYPE I. DIVISION OF A QUANTITY.

1. These diagrams show what William and Louise did with the money which they earned last summer. Make a table showing what each of them did with the money. What percent of his money did William put in the bank?

2. What percent did he spend for clothes? For books? For tools?

3. What percent did Louise spend for clothes? For books? For music?

TYPE II. COMPARISON OF MAGNITUDES.

1. According to the diagram how many pounds can Charles lift? How many can Dick lift? Fred? Tom?

2. Draw a diagram to show how many of the exercises in the arithmetic practice on page 165 each of these children did correctly, in 5 min.

Alice, who had 30 correct
Anna, who had 19 correct
Nell, who had 24 correct
Sarah, who had 36 correct

Let ⅛ in. of distance up and down equal 1 exercise correct, 1 in. equal 8 exercises correct, etc.

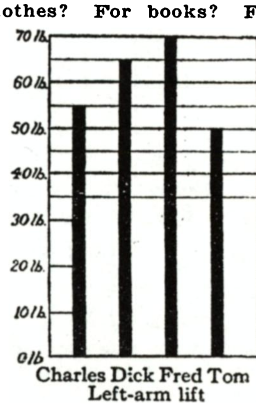
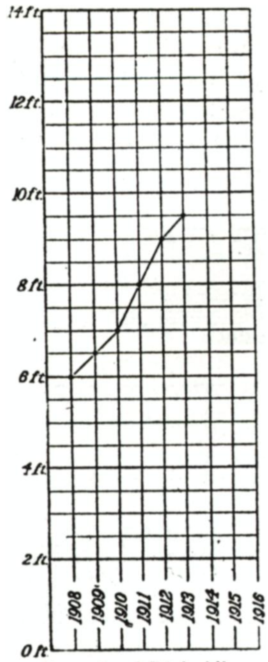


FIGURE 2



Records of Dick Allen in the broad jump

FIGURE 3

TYPE III.

COMPARISON OF ORDERED MAGNITUDES

3. Using thin paper trace and complete this diagram or graph, which tells how Dick Allen improved in the broad jump. His records were 8 ft. in 1911, 9 ft. in 1912, 9 ft. 6 in. in 1913, 10 ft. 6 in. in 1914, 13 ft. in 1915, and 13 ft. 6 in. in 1916.

4. Draw a diagram or graph to show how Elsie improved in repeated trials with the practice test on page 165.

Her scores in the 10 successive weeks were 17, 17, 19, 23, 22, 23, 24, 25, 24, 26.

5. Do the practice of page 165 twice a day for five days. Draw a graph showing how well you did in each trial and how much you improved.

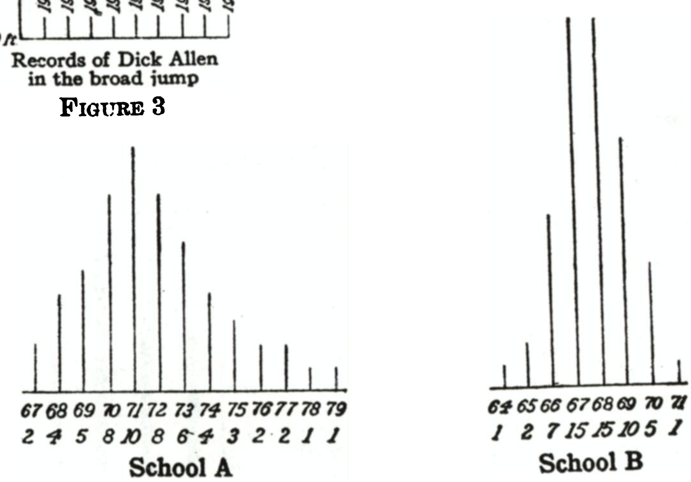


FIGURE 4

TYPE III-A. GRAPHS OF RELATIVE FREQUENCIES.

5. Examine the diagrams on p. 196. Read the diagram for School A, saying, "On 2 days, the temperature was 67°; on 4 days it was 68°; on 5 days it was 69°"; etc. Find what percent of the days were "satisfactory" as to temperature in School A. In School B. (A temperature of 66, 67, 68, 69 or 70 is called "satisfactory.")

Figure 1 is an illustration of graphic descriptions of the way in which a certain quantity is divided, as when we wish to show how a family spends its income, or how the population of a country is divided with respect to race or how a pupil spends the day.

Figure 2 is an illustration of general comparisons of magnitudes, as of the number of boys and of girls in a school, or of the values of certain products of a farm, or of the number of voters of each political affiliation, or of the size of a school ten years ago and now, or the percent of illiterary in each of forty states.

When the magnitudes to be compared form a series easy to put in order by their relation to some characteristic, this is usually done. If, for example, we have the size of a school now, five years ago, ten years ago, fifteen years ago, etc., it is obviously best to put the columns or bars showing size in a chronological order. If we have the number of children at each age in a school, it is obviously best to put the columns or bars in the order: aged 5, aged 6, aged 7, etc.

Figures 3 and 4 illustrate such graphs of class III. The typical graph of the relation of one variable to another is the most notable case of class III. Graphs of surfaces of frequency or distribution (III-A) are a group of increasing importance in the social sciences.

It is customary to draw a distinction between statistical graphs and mathematical graphs. The former include all of classes I and II and those of class III which are not readily analyzable into regular relations conveniently expressed in the relation of y to x. This distinction is not sharp or rigid, some weight being also given to custom and convenience.

Return now to our question A, "Shall school algebra deal with graphs of classes I and II, or shall it limit itself to class III?" The future will probably save us the trouble of answering, because acquaintance with graphs of classes I and II, and with simple cases in class III, will probably be given in grades 4 to 8. Figures 1, 2, 3, and 4 are, in fact, all from a recent textbook in arithmetic. When it has not been given already, such work may deserve a place in ninth grade mathematics because

of its intrinsic worth, and because of the interest it lends to the graphic presentation of the relations between two variables.

Graphs of class I and class II are, however, psychologically useful in algebra only as possible introductions to those of class III. For pupils at the high school level, the erection of a series of columns and the formation of the curve joining the mid-points of their tops, seems adequate. Also it seems to the writer that the development of a serial graph from the graph of mere comparison will introduce more interferences than aids to comprehension of the former. Logically, the graph of a systematic, ordered relation may be thought of as a sub-class of, and development from, the graph comparing any quantities—the population of states, or the scores of ball players, or the heights of species of trees. But it seems sounder psychology to teach the systematic graph, ordered in relation to time, or number of articles, or size of the object, or force exerted, etc., by itself.*

The elementary facts of surfaces of frequency deserve a place in the curriculum somewhere in grades 6 to 9 (preferably in grades 7 or 8), the pupil being taught to understand such graphs and to construct them from the tabular data. Their relation to the curve $y = e^{-x^2}$, and to the coefficients of the binomial expansion may perhaps be shown toward the end of a course in algebra.† Their consideration along with the graphs of simpler relations of one variable to another will be confusing and should be avoided.

Our second question was: Assuming that algebra in grades 9 and 10 limits itself to "mathematical" graphs of the relation of one variable to another and to such introductory matter as is of value therewith, shall it deal with graphs not presentable in any equations that the pupil can master, or shall it restrict itself to the straight line, rectangular hyperbola, parabola, circle, and the like?" It may be answered provisionally as follows: Enough work with irregular relations should be given so that the pupil will appreciate the regularity of regular ones by contrast, and also the place of these regular relation lines amongst relation lines in general. Except for that, it seems best to spend the time on standard forms whose mathematical significance can

* That is, to make the principle of organization the facts and laws to be expressed, rather than the general nature of graphic presentations.
† More suitably, probably, in an advanced course.

be realized.* The curves for $y = ax$, $y = ax + b$, $y = x^2$, $y = x^2 + a$, $y = x^3$, $y = x^3 + a$, $y = \frac{a}{x}$, $y = x^2 + ax + b$, $y^2 + x^2 = a^2$, and $y = a^x$ will probably provide sufficient variety to make the principles general.

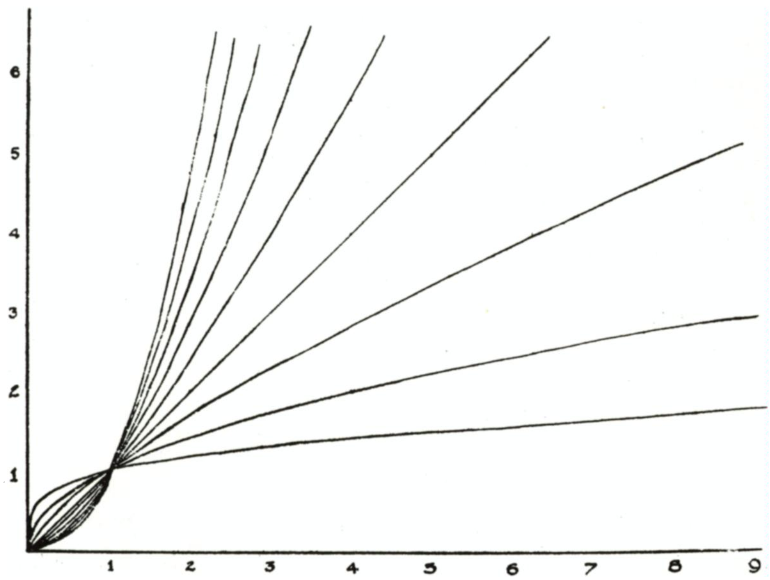


FIGURE 5

When the topic of fractional exponents is taken up, a review and extension of the graphic treatment of regular relations may be had by presenting the curves for $y = x^{\frac{1}{2}}$, $y = x^{\frac{2}{3}}$, $y = x^{\frac{3}{4}}$ $y = x^{\frac{3}{2}}$, $y = x^{\frac{4}{3}}$, and the like, as in Figure 5. If logarithms are taught, the construction of such curves provides excellent exercises in computation and approximations, and a useful application of knowledge of fractional exponents. Simple facts about $y = k^x$ may be taught also.

---

* Including, perhaps, standard forms obscured by chance errors which disappear in a coarser grouping of the data.